Magnetotransport in inhomogeneous magnetic fields

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(Dated: February 2, 2008)

Quantum transport in inhomogeneous magnetic fields is investigated numerically in two-dimensional systems using the equation of motion method. In particular, the diffusion of electrons in random magnetic fields in the presence of additional weak uniform magnetic fields is examined. It is found that the conductivity is strongly suppressed by the additional uniform magnetic field and saturates when the uniform magnetic field becomes on the order of the fluctuation of the random magnetic field. The value of the conductivity at this saturation is found to be insensitive to the magnitude of the fluctuation of the random field. The effect of random potential on the magnetoconductance is also discussed.

I. INTRODUCTION

Quantum transport in two-dimensional (2D) disordered systems in inhomogeneous magnetic fields has been studied extensively, in connection with the fractional quantum Hall effect (FQHE).[1] In particular, the Anderson localization[2] in a 2D system in random magnetic fields(RMF) with zero mean, which arises in the mean field theory of the fractional quantum Hall effect at filling factor $\nu = 1/2$, has been studied by many authors.[3,4,5,6,7,8,9,10,11] From the theoretical point of view, whether such a system has a metallic phase or not has been an important issue. Systems in random magnetic fields belong to the 2D unitary universality class which is, in general, expected to have no metallic phase. If the system has a metallic phase, it throws a question on the validity of the conventional classification of the universality classes of the Anderson transition. Although many numerical as well as analytical attempts has been made to clarify this point, the conclusions are so far still controversial. On the other hand, the singular behavior of the conductance fluctuations[12] and the density of states at the band center ($E = 0$) has been investigated and argued that these singularities are governed by the chiral unitary universality class.[13,14]

Apart from such a theoretical interest, the transport properties of 2D electron systems in random magnetic fields are important to understand the experiments in the quantum Hall systems. In fact, the structure observed in the magnetoresistance at the filling factor 1/2 has been analyzed using models with random magnetic fields.[15] On the other hand, recently, two-dimensional electron systems in random magnetic fields have been realized experimentally by attaching small magnets on the layer parallel to the 2D electron gas in a semiconductor heterostructure.[16] In such a system, one can also control the strength of the random magnetic fields. The magnetotransport has been measured and interestingly, it is found that the magnetoresistance in systems having random magnetic fields exhibits similar structures to that in the fractional quantum Hall system at the filling factor $\nu = 1/2$.[16] Since the origins of the magnetic field fluctuations in these two systems are quite different, it is important to clarify the origin of this similarity in the magnetoresistance between these two systems.

One of the significant features of the magnetoresistance observed in the recent experiment[16] is the increase of the magnetoresistance in two different scales of the magnetic field. One is the dip structure around zero field whose width is on the order of the fluctuation of the random field, and the other is the positive magnetoresistance in a larger scale of the magnetic field followed by the Shubnikov-de Haas oscillations. In recent years, the semi-classical theory for the magnetotransport in a smoothly varying random magnetic field has been developed,[17] where the snake state near zero magnetic field lines plays an important role. In the case of a weak disorder, a pronounced positive magnetoresistance coming from a classical origin has been reported for small magnetic fields.[17] Although the semi-classical approach provides qualitatively consistent results with the experiments, it is also important to examine the transport property by the quantum mechanical calculations for its further understanding.

In the present paper, we study the magnetoconductance in the random magnetic field in two dimensions. We consider a tight-binding model in random fluxes having no spatial correlation, which has been analyzed by many authors for the study of the Anderson localization in random magnetic fields. In two dimensions, it has been shown that such a model has the large localization length near the band center and the diffusive behavior is observed in that energy regime for numerically accessible length scales.[11] Since there is no spatial correlation in random magnetic fields, the random magnetic field is not smooth and hence the validity of the semi-classical concepts is not obvious for the present model. In our model, we find that the conductivity is strongly suppressed by adding a uniform field and also that it shows a saturation when the uniform field becomes on the order of the fluctuation of the random magnetic field.

We adopt the equation of motion method in order to examine numerically the diffusion of an electron in inhomogeneous magnetic fields. Adopting this method, we have an advantage that very large systems compared with other numerical methods can be considered. However, within the present method, we are able to estimate the longitudinal conductivity (σ_{xx}) only, and not the Hall conductivity (σ_{xy}). Due to this, we are not able to discuss the magnetoresistance directly. As mentioned above, the 2D random magnetic field system has been shown to have very large localization length near the band center. It is then natural to assume that, in the present model, this regime is responsible for the metallic behavior observed in experiments. We therefore confine ourselves to the case that the Fermi energy lies near the band center. First, we discuss the transport in the random magnetic fields with zero mean and next, we examine the effect of additional uniform magnetic fields. The strength of the additional uniform magnetic field considered in the present study is assumed to be on the same order as or smaller than that of the random magnetic field. Implications from our numerical results for the magnetoconductance are discussed in comparison with the recent experimental results.

II. MODEL AND METHOD

In order to describe the two-dimensional system in random magnetic fields, we consider the following Hamiltonian

$$H = \sum_{<i,j>} V \exp(\mathrm{i}\theta_{i,j}) C_i^\dagger C_j + \sum_i \varepsilon_i C_i^\dagger C_i \qquad (1)$$

on the square lattice. Here $C_i^\dagger(C_i)$ denotes the creation(annihilation) operator of an electron on the site i and $\{\varepsilon_i\}$ denote the random potential distributed independently in the range $[-W/2, W/2]$. The phases $\{\theta_{i,j}\}$ are related to the magnetic fluxes $\{\phi_i\}$ through the plaquette $(i, i+\hat{x}, i+\hat{x}+\hat{y}, i+\hat{y})$ as

$$\theta_{i,i+\hat{x}} + \theta_{i+\hat{x},i+\hat{x}+\hat{y}} + \theta_{i+\hat{x}+\hat{y},i+\hat{y}} + \theta_{i+\hat{y},i} = -2\pi\phi_i/\phi_0 \quad (2)$$

where $\phi_0 = h/|e|$ stands for the unit flux. The fluxes $\{\phi_i\}$ are also assumed to be distributed independently in each plaquette. The probability distribution $P(\phi)$ of the flux ϕ is given by

$$P(\phi) = \begin{cases} 1/h_{\mathrm{rf}} & \text{for} \quad |\phi/\phi_0| \le h_{\mathrm{rf}}/2 \\ 0 & \text{otherwise} \end{cases} . \qquad (3)$$

The variance of the distribution is accordingly given by

$$\langle \phi_i \phi_j \rangle = \frac{h_{\mathrm{rf}}^2}{12} \phi_0^2 \delta_{i,j}. \qquad (4)$$

In order to solve the time-dependent Schrödinger equation numerically, we employ the decomposition formula for exponential operators.[18] The basic formula used in the present paper is the forth order formula

$$\exp(x[A_1 + \cdots + A_n]) = S(xp)^2 S(x(1-4p)) S(xp)^2 + O(x^5), \qquad (5)$$

where

$$S(x) = \mathrm{e}^{xA_1/2} \cdots \mathrm{e}^{xA_{n-1}/2} \mathrm{e}^{xA_n} \mathrm{e}^{xA_{n-1}/2} \cdots \mathrm{e}^{xA_1/2}. \quad (6)$$

The parameter p is given by $p = (4 - 4^{1/3})^{-1}$ and $A_1, \ldots, A_n$ are arbitrary operators. We divide the Hamiltonian into five parts as in the previous paper[11] so that each part is represented as the direct product of 2×2 matrices. By applying this formula to the time evolution operator $U(t) \equiv \exp(-\mathrm{i}Ht/\hbar)$, we obtain

$$\begin{aligned} U(\delta t) &= U_2(-\mathrm{i}p\delta t/\hbar)^2 U_2(-\mathrm{i}(1-4p)\delta t/\hbar) U_2(-\mathrm{i}p\delta t/\hbar)^2 && (7) \\ &\quad + O(\delta t^5) && (8) \end{aligned}$$

with

$$U_2(x) = \mathrm{e}^{xH_1/2} \cdots \mathrm{e}^{xH_4/2} \mathrm{e}^{xH_5} \mathrm{e}^{xH_4/2} \cdots \mathrm{e}^{xH_1/2}, \qquad (9)$$

where $H = H_1 + \cdots + H_5$. It is to be noted that the U_2 can be expressed in an analytical form while the original evolution operator U can not be evaluated exactly without performing the exact diagonalization of the whole system. We are thus able to consider larger system-sizes than other numerical methods, such as exact diagonalization and the recursion method based on the Landauer formula, which is one of the advantages of the present method. This method has already been successfully applied to the case of $W = 0$ and $h_{\mathrm{rf}} = 1$[11] as well as to the 2D symplectic class.[19]

The system we consider is the square lattice of the size 999×999 with the fixed boundary condition. All the length scales are measured in units of the lattice constant a. To prepare the initial wave packet with energy E, we numerically diagonalize the subsystem (21×21) located at the center of the whole system and take the eigenstate whose eigenvalue is the closest to E as the initial wave function. In the following, we set the energy E to be $E/V = -0.5$ which is close to the band center. The single time step δt is set to be $\delta t = 0.02\hbar/V$. With this time step, the fluctuations of the expectation value of the Hamiltonian is safely neglected throughout the present simulation ($t \le 200\hbar/V$). We observe the second moment defined by

$$\langle \mathbf{r}^2(t) \rangle_c \equiv \langle \mathbf{r}^2(t) \rangle - \langle \mathbf{r}(t) \rangle^2 \qquad (10)$$

with

$$\langle \mathbf{r}^n(t) \rangle = \sum_{\mathbf{r}} \mathbf{r}^n |\psi(\mathbf{r}, t)|^2, \qquad (n = 1, 2) \qquad (11)$$

where $\psi(\mathbf{r}, t)$ denotes the wave function at time t. In the diffusive regime, the second moment is expected to grow in proportion to t

$$\langle r^2 \rangle_c = 2dDt, \qquad (12)$$

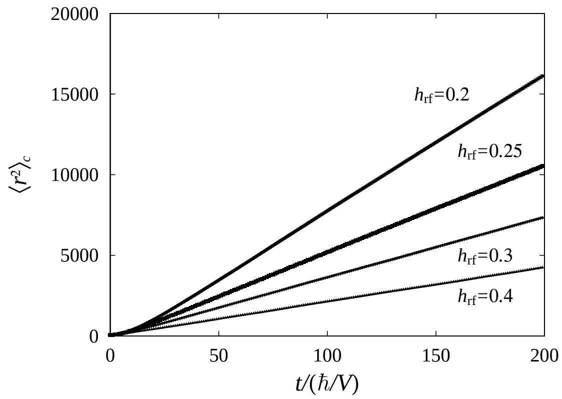

FIG. 1: The second moment as a function of time for $h_{\rm rf} = 0.2, 0.25, 0.3$, and 0.4. Five realizations of random magnetic fields are considered for each $h_{\rm rf}$.

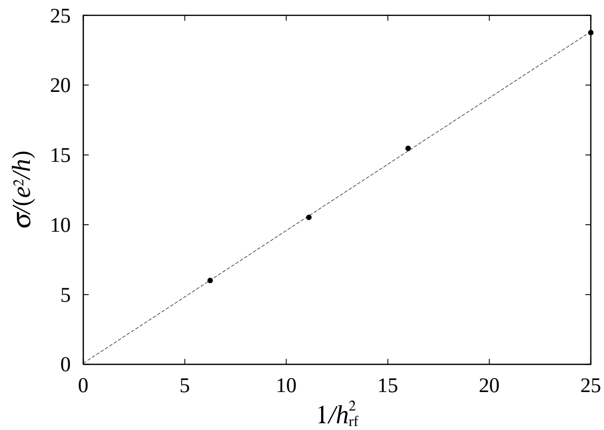

FIG. 2: Conductivity in units of e^2/h for $h_{\rm rf} = 0.2$, 0.25, 0.3 and 0.4.

where the diffusion coefficient is denoted by D and d is the dimensionality of the system. The diffusion coefficient D is related to the conductivity by the Einstein relation $\sigma = e^2 D\rho$.[20] Here ρ denotes the density of states. In Fig. 1, the second moment is shown as a function of time for the case of $W = 0$. For each value of $h_{\rm rf}$, five realizations of random magnetic fields are considered. It is clearly seen that for $t \geq 50\hbar/V$ the second moment increases in proportion to t, which means that the system is in the diffusive regime. For $t \leq 50\hbar/V$, we see the ballistic behavior, where the second moment grows as $\propto t^2$. By examining the behavior of the second moment, we can clearly distinguish whether the system is in the diffusive regime or in the ballistic regime. To estimate the diffusion coefficient we discard the data in the ballistic regime. We also do not consider the time scale where the wave packet reaches to the edge of the whole system. Within the time scale considered in the present paper $t \leq 200\hbar/V$, we do not observe any sign of the saturation of the second moment due to the finiteness of the system. To obtain the conductivity, we need the density of states too. We estimate it by the Green function method[21] for strips the width of which is $12 \leq M \leq 30$.

III. NUMERICAL RESULT

Let us first discuss how the diffusion of an electron in random magnetic fields depends on the strength of the fluctuation of random magnetic fields. We estimate the diffusion coefficients from the behavior of the second moment shown in Fig. 1. It is clearly seen that the diffusion coefficient becomes smaller as the fluctuation of the random magnetic fields increases. The density of states at $E = -0.5V$ is estimated to be $\rho \approx 0.178$, 0.181, 0.179, and $0.179(1/a^2V)$ for $h_{\rm rf} = 0.2$, 0.25, 0.3 and 0.4, respectively. It depends on the strength of the RMF very weakly near the band center. On the other hand, the diffusion coefficient D depends strongly on the strength of the RMF which is estimated to be $4D = 85.22 \pm 0.03$, 54.44 ± 0.04, 37.40 ± 0.03 and $21.30 \pm 0.02 (a^2V/\hbar)$ for $h_{\rm rf} = 0.2$, 0.25, 0.3 and 0.4, respectively. In Fig. 2, the conductivity, evaluated from these diffusion coefficients and the density of states, is plotted for small values of $h_{\rm rf}$. Here we clearly see that the conductivity is inversely proportional to the square of $h_{\rm rf}$, namely, $\sigma \propto 1/h_{\rm rf}^2$. This fact indicates that the present regime would be well described by the Born approximation.[15] In the recent experiment,[16] it is observed that the relative change of the resistivity by the random magnetic fields is in proportion to the square of the fluctuation of the random magnetic fields, which is qualitatively consistent with the present results (Fig. 2).

Next, we apply an additional uniform magnetic field and examine its effect on the electron diffusion. The uniform magnetic field through each plaquette of the square lattice is denoted by $\phi_{\rm uni}$. The total magnetic field per plaquette is then $\phi_{\rm uni} + \phi$, where $\phi_{\rm uni}$ is common to all the plaquette and ϕ is distributed independently as (3). The second moments for $h_{\rm rf} = 0.2$ under various values of the uniform magnetic fields are shown in Fig. 3. Here it is clear that the diffusion of electrons is strongly suppressed by the additional uniform magnetic field. Note that the strength of the uniform field considered is smaller than the fluctuation of the random field $\sqrt{\langle \phi^2/\phi_0^2 \rangle} = h_{\rm rf}/\sqrt{12} = 0.0577\ldots$. The diffusion coefficients and the density of states estimated for various values of the uniform fields are summarized in Table 1. Here we see again that the density of states is not sensitive to the uniform magnetic field. We also perform numerical calculations for $h_{\rm rf} = 0.3$ and 0.4. In Fig. 4, the estimated conductivity is plotted as a function of the uniform field scaled by $\sqrt{\langle \phi^2 \rangle}$. For these values of the random magnetic field, it is commonly observed that the negative magnetoconductance occurs when the uniform field is weaker than the fluctuation of the random field. It is also observed that at $\phi_{\rm uni} \approx \sqrt{\langle \phi^2 \rangle}$, the conductivity takes a value on the order of e^2/h and is, interestingly, in-

TABLE I: The diffusion coefficients D and the density of states ρ for $h_{\mathrm{rf}} = 0.2$

$\phi_{\mathrm{uni}}/\phi_0$	$4D$	ρ
0	85.22 ± 0.03	0.178
0.002	83.25 ± 0.03	0.179
0.005	76.37 ± 0.04	0.179
0.01	59.54 ± 0.02	0.178
0.02	34.61 ± 0.04	0.179
0.04	14.65 ± 0.02	0.183
0.06	8.69 ± 0.02	0.189
0.08	6.69 ± 0.02	0.177
0.1	5.31 ± 0.02	0.168

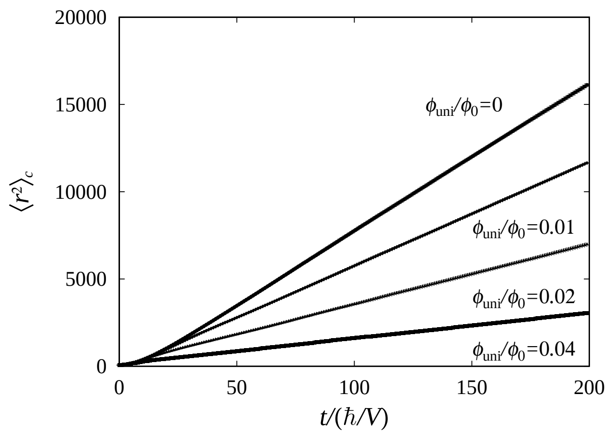

FIG. 3: Electron diffusion for $\phi_{\mathrm{uni}}/\phi_0 = 0$, 0.01, 0.02, and 0.04, where ϕ_{uni} stands for the strength of the uniform flux per plaquette.

sensitive to the magnitude of the random magnetic field.

In order to see whether this peak structure around zero field can be observed in the presence of the random potential ($W \neq 0$), we also examine the electron diffusion with the additional random potential. We consider the case where $h_{\mathrm{rf}} = 0.2$ and $W = 0$, 1 and 2 and find that the peak structure disappears at $W = 2$ (Fig. 5). The random potential $W = 2$ on the square lattice yields the mean free path on the order of 4 lattice constants.[22] This result indicates that to observe this enhancement of conductivity at zero uniform field we need fairly clean samples.

Estimation of the mean free path in the random magnetic field, especially for $W = 0$, is a subtle problem.[23] If we define the relaxation time for the transport[23] by $\tau_{\mathrm{tr}} = 2D/v_{\mathrm{F}}^2$, where v_{F} denotes the Fermi velocity, the mean free path l_{rf} in the random magnetic field can be defined by $l_{\mathrm{rf}} = v_{\mathrm{F}}\tau_{\mathrm{tr}} = 2D/v_{\mathrm{F}}$. Near the band center, the average Fermi velocity can be estimated to be on the order of $2aV/\hbar$. The mean free path l_{rf} is then evaluated to be about $21a$, $9a$, and $5a$ for $h_{\mathrm{rf}} = 0.2$, 0.3 and 0.4, respectively.

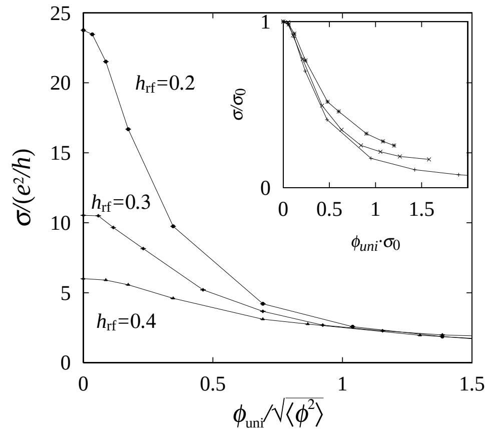

FIG. 4: Magnetoconductance for $h_{\mathrm{rf}} = 0.2$, 0.3 and 0.4. The uniform magnetic flux ϕ_{uni} is scaled by the strength of the random magnetic flux $\sqrt{\langle\phi^2\rangle}$. Inset: The normalized conductivity σ/σ_0 plotted as a function of $\phi_{\mathrm{uni}} \cdot \sigma_0$. From the top, results for $h_{\mathrm{rf}} = 0.4$, 0.3 and 0.2 are presented. The normalized data do not fall on the one curve, suggesting the deviation from the Drude behavior.

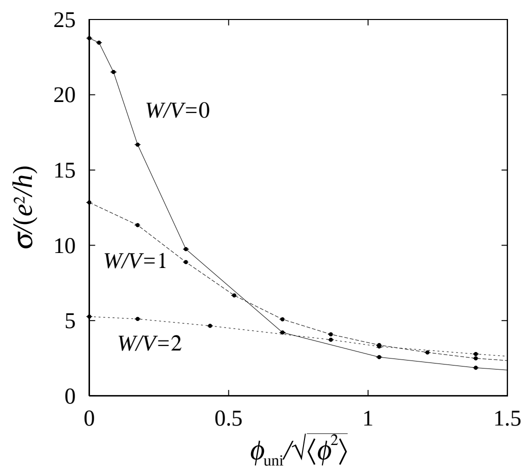

FIG. 5: Conductivity in units of e^2/h for $h_{\mathrm{rf}} = 0.2$ and $W/V = 0, 1$ and 2.

IV. DISCUSSION

We have demonstrated that in the absence of the uniform magnetic field, the conductivity is inversely proportional to the square of the magnitude of the random field, and hence is very sensitive to the strength of the random magnetic field. In contrast, we have also shown that the conductivity takes a value on the order of e^2/h and is insensitive to the strength of the random magnetic field if the uniform field is set to be $\phi_{\mathrm{uni}} \approx \sqrt{\langle\phi^2\rangle}$ (Fig. 4). These two properties yield the peak structure at zero field, especially, in the case of the weak random magnetic fields.

It may be useful to discuss our results in view of the Drude formula $\sigma = \sigma_0/(1 + \omega_c^2\tau^2)$, where ω_c and τ are

the cyclotron frequency and the relaxation time, respectively. Our results indicate that $\sigma_0 \propto \tau \propto \langle\phi^2\rangle^{-1}$(Fig. 2). The Drude theory then yields the form $\sigma/\sigma_0 = 1/(1 + A(\phi_{\rm uni}\cdot\sigma_0)^2)$, where A is a constant fixed by the density of electrons and independent of $\phi_{\rm uni}$ and ϕ. We have found that although the data for small magnetic fields $\phi_{\rm uni} \ll \sqrt{\langle\phi^2\rangle}$ seem to be consistent with this scaling form, it is unlikely that the Drude formula accounts for the behavior around $\phi_{\rm uni} \approx \sqrt{\langle\phi^2\rangle}$. The deviation from the Drude theory, in which the resistivity is independent of the magnetic field, comes from the fact that the conductivity is insensitive to the magnitude of the random magnetic field at $\phi_{\rm uni} \approx \sqrt{\langle\phi^2\rangle}$ (Fig.4).

Let us consider this insensitivity of the conductivity to the magnitude of the random magnetic field at $\phi_{\rm uni} \approx \sqrt{\langle\phi^2\rangle}$ in more detail. With this condition $\phi_{\rm uni} \approx \sqrt{\langle\phi^2\rangle}$, the system is almost equivalent to a system having the random magnetic fields distributed in the range $0 \le \phi/\phi_0 \le h$ and having no uniform magnetic field $\phi_{\rm uni} = 0$. We have then evaluated the conductivity in such a system for 0.2, 0.3, 0.4 and 0.5 and, indeed, found that for all these values of h the conductivity is insensitive to the value of h and falls in the range $1.4 \sim 1.9(e^2/h)$. This would be one of the significant transport properties specific to the random magnetic fields.

In order to consider a possible relationship to the experiments, it may be useful to identify the correlation length of the random fields in experiments with the lattice constant of the present model. Our calculation then implies that the sample having mean free paths longer than 4 correlation lengths of the random fields is very sensitive to the application of uniform magnetic fields. This implication seems to be consistent with the experiments[16] where the dip structure is observed in samples having a mean free path larger than the correlation length of the random magnetic field.

In summary, we have investigated the electron diffusion in the random magnetic field with additional uniform magnetic fields by the equation of motion method. We have found that a sharp peak at zero uniform field appears in magnetoconductance in the absence of the spatial correlation of random magnetic fields, where the semi-classical theory can not be applied. The width of the peak turns out to be on the order of the fluctuation of the random magnetic field. The conductivity at $\phi_{\rm uni} \approx \sqrt{\langle\phi^2\rangle}$ is found to be insensitive to the magnitude of the random magnetic field. This peak structure disappears when the mean free path becomes shorter by introducing the random potential. Although the present method enables us to simulate very large systems, we can obtain the longitudinal conductivity σ_{xx} only. Detailed analysis of σ_{xx} in experiments is required to examine the relevance of the present results.

Acknowledgments

The authors thank M. Ando, Y.Ono, B.Kramer and S. Kettemann for valuable discussions. Numerical calculations were performed by the facilities of the Supercomputer Center, Institute for Solid State Physics, University of Tokyo. This work is partly supported by the Grants-in-aid No.13740244 from Japan Society for the Promotion of Science.

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